B.A./B.Sc. 1st Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

Paper : BMHI-CC-II (Algebra)

Time: 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meaning.

Group-A

1. Answer *any ten* questions from the following: 2×10=20

(a) Transform $x^3 - 6x^2 + 5x + 12 = 0$ into a equation lacking the second degree term.

(b) If a, b, c be all real numbers, prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(c) If $a^n - 1$ is prime number for some positive integer n, prove that $a = 2$.

(d) Find arg$z$, where $z = 1 + \cos 2\theta + i\sin 2\theta$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

(e) Let $A = \left\{x \in \mathbb{R} \mid x \neq \frac{1}{2}\right\}$ and define $f : A \to \mathbb{R}$ by $f(x) = \frac{4x}{2x-1}$ $\forall$ $x \in A$. Is f one to one? Justify your answer.

(f) Given three consecutive integers $a, a+1, a+2$. Prove that exactly one of them is divisible by 3.

(g) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one to one but not onto. Justification needed.

(h) Let $\rho = \left\{(a, b) \mid a, b \in \mathbb{N}, \frac{a}{b} \text{ is an integer}\right\}$ be a binary relation on $\mathbb{N}$. Is ρ equivalence relation? Justify your answer.

(i) Let ρ denote an equivalence relation on set A. Let $a \in A$, prove that for any $x \in A$, $x\,\rho\,a$ iff $\mathrm{cl}(x) = \mathrm{cl}(a)$.

(j) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then show that T is one to one if and only if the equation $T(x) = 0$ has only the trivial solution.

(k) What is the Geometric object corresponding to smallest subspace V_o containing a non-zero vector $u = (x, y)$ in $\mathbb{R}^2$? Justify your answer.

(l) Prove that $(n+1)^n \geq 2^n.\underline{|n}$, where n is any positive integer.

(m) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 2 \end{pmatrix}$. Find the eigenvalues of the matrix $A^5 - I_3$.

(n) Find all values of k such that the set $S = \{(k,1,k),(0,k,1),(1,1,1)\}$ form a basis of $\mathbb{R}^3$. (Give reasons).

(o) Prove that there are infinitely many primes of form $4q+3, q \in \mathbb{Z}$.

Group-B

2. Answer *any four* questions from the following: 5×4=20

(a) (i) Prove that $\arg z - \arg(-z) = \pm\pi$ according to $\arg z > 0$ or $\arg z < 0$.

(ii) If a and b are relatively prime integers, prove that $gcd\ (a+b, a-b) = 1$ or 2. 3+2=5

(b) (i) For (x,y) and (u,v) in $\mathbb{R}^2$, define $(x,y)\,\rho\,(u,v)$ if and only if $x^2+y^2 = u^2+v^2$. Prove that ρ is an equivalence relation on $\mathbb{R}^2$ and interpret the equivalence classes geometrically.

(ii) Suppose a and b are integers and $3|(a^2+b^2)$, show that $3|a$ and $3|b$. 3+2=5

(c) (i) If n is a positive integer, prove that $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$.

(ii) If $k \in \mathbb{N}$, Prove that $gcd(3k+2, 5k+3) = 1$. 3+2=5

(d) (i) By using Sturm's method, find the positions of the real roots of the equation

$x^4 - 2x^3 + 7x^2 + 10x + 10 = 0$.

(ii) Find the number of reflexive relations on a set of 3 elements. 3+2=10

(e) (i) Suppose λ is an eigenvalue of a real symmetric matrix A. Prove that $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$.

(ii) Give an example of a linear transformation $T : R^2 \to R^2$ such that $T^2(\alpha) = -\alpha$. 3+2=5

(f) (i) Let $V = \mathbb{R}^n$ and A be a $n \times n$ natrix. If $AX = 0$ has a unique solution, prove that $AX = b$ has a unique solution for every $b \in \mathbb{R}^n$.

(ii) Prove that for all integers $n > 1$, $n^4 + 4$ is composite. 3+2=5

Group-C

3. Answer *any two* questions from the following: 10×2=20

(a) (i) State and prove De Moivre's theorem.

(ii) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. 5+5=10

(b) (i) Solve the system of equations.

$$x - 2y + z - t = 0$$
$$x + y - 2z + 3t = 0$$
$$4x + y - 5z + 8t = 0$$
$$5x - 7y + 2z - t = 0$$

(ii) If u and v are two integers and $v > 0$, prove that there exist two unique integers s and t such that $u = sv + t$ with $0 \leq t < v$.

(iii) Find the equation whose roots are $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$ where α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$. 4+4+4+2=10

(c) (i) Prove that the square of an odd integer is of the form $8k + 1$, where k is an integer.

(ii) Solve the equation: $x^4 - 9x^3 + 28x^2 - 38x + 24 = 0$.

(iii) Find the range space, null space, rank of T and nullity of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x, y, z, t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t)$.

2+4+(1+1+1+1)=10

(d) (i) Prove that eigenvalues of a real symmetric matrix are real.

(ii) Show that $x^n - 1 = (x - 1)\prod_{k=1}^{\frac{1}{2}(n-1)}\left(x^2 - 2x\cos\frac{2k\pi}{n} + 1\right)$, where n is an odd integer.

(iii) Let $T : V \to W$ be a linear transformation. Prove that T is $1 - 1$ if and only if it maps any linearly independent set of V to a linearly independent set of W. 3+4+3 =10
