PROGRAMME: B.Sc. PHYSICS HONOURS

SEMESTER - V

Course: CLASSICAL DYNAMICS (DSE-2)

Lecture Note: 2

[redacted]

Generalized Co-ordinate :⇒

It is the number of indepen- ~~dt~~ dent co-ordinate necessary to describe the configaration of a dynamical System. For a system of 'f' number of degree of freedom having holonomic constraint the number of independent co-ordinate (generalized) is equal to the number of degree of freedom.

These co-ordinate are generally denoted as $q_1, q_2, q_3 \cdots\cdots q_f = \{q_j\}$

where $j = 1, 2, 3 \cdots\cdots f$.

It may or may not have the dimension of length.

The position vector of the ith particle

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3 \cdots\cdots q_f, t)$$

Scleronomic

For holonomice constraint $\vec{r}_i$ is not explicitly function of time t

i.e. $\boxed{\dfrac{\partial \vec{r}_i}{\partial t} = 0}$

Priciple of Virtual Work :-

Any imaginary displacement which is consistent with the constraint relation at a given instant is called **Virtual ~~wr~~ displacement.**

Let us consider a system of N particles moving from one configaration space, defined by co-ordinates to another configaration at a given instant of time. i.e. no real change in time, consisted with the external forces and the constraint imposed on the system, is then imagine to be displaced by an infintesmial vector $\delta\vec{r_i}$ which called virtual displacement. It is different form of actual displacement $d\vec{r_i}$ which occurs in a time interval dt during which external forces and constraint may change

$$\delta\vec{r_i} = d\vec{r_i}\Big|_{dt=0}$$

For a system to be in equilibrium the resultant force acting on ~~the~~ each particle must be zero.

$$\vec{F_i} = 0$$

$\vec{F_i}$ being the force on ith particle.

The virtual work done by the force $\vec{F_i}$ due to virtual displacement $\delta\vec{r_i}$ of the ith particle

$$\vec{F_i} \cdot \delta\vec{r_i} = 0 \quad \text{for equilibrium}$$

For all the particles of the system the condition of equilibrium is then

$$\sum_i \vec{F_i} \cdot \delta\vec{r_i} = 0$$

But $\vec{F_i} = \vec{F_i}^{(a)} + \vec{f_i}$

where $\vec{F_i}^{(a)}$ is the applied external force and $\vec{f_i}$ is the force due to Constraint.

In most of the case the force due to constraint does no work. (It is not true for frictional forces)

$$\therefore \quad \vec{f_i} \cdot \delta\vec{r_i} = 0$$

So the condition of equilibrium is then

$$\delta W = \sum_i \vec{F_i}^{(a)} \cdot \delta\vec{r_i} = 0$$

It should be noted that $\delta\vec{r_i}$ is not independent but connected by force due to constraint.

Thus a system to be equilibrium if and only if the total virtual work done by the applied forces on the particle is zero, provided there is no frictional force. This is known as Principle of Virtual Work.

D' Alembert's Principle :—

The principle of virtual work done may be extended for the case of dynamics. The equation of motion of the ith particle

$$\vec{F_i} = \frac{d\vec{p_i}}{dt} = \dot{\vec{p_i}}$$

Where $\vec{p_i}$ is the linear momentum of the ith particle.

$$\therefore \quad \vec{F_i} + (-\dot{\vec{p_i}}) = 0$$

Thus a moving particle will be in equilibrium under the force $\vec{F_i} + (-\dot{\vec{p_i}}) = 0$ i.e. under the actual force $\vec{F_i}$ plus a reverse effective force $(-\dot{\vec{p_i}})$. In this way the dynamics is reduce to static. Hence the condition of equilibrium for a system of particles in dynamics is

$$\sum_i (\vec{F_i} - \dot{\vec{p_i}}) \cdot \delta\vec{r_i} = 0$$

$$\vec{F_i} = \vec{F_i}^{(a)} + \vec{f_i}$$

If the force due to constraint does no work, then the condition of equilibrium in dynamics

$$\sum (\vec{F_i}^{(a)} - \dot{\vec{p_i}}) \cdot \delta\vec{r_i} = 0$$

This is called D'Alembert Principle.

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It states that the total work done by the applied and reverse effective force due to a virtual ~~code~~ displacement $\delta\vec{r_i}$ is zero.

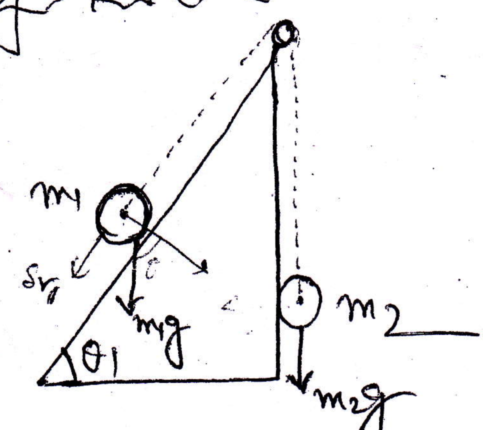
Advantages:-

i) It does not involve force due to Constraint.

ii) It is true for Scleronomic and rheonomic constraints.

Some application of D'Alembert Principle:-

i) Single inclined plane:-



Let two masses m_1 and m_2 are moving in a single inclined plane as shown in fig. They are joined by inextensible string of length l. Let $\vec{r_1}$ and $\vec{r_2}$ are the position vector of two masses with respect to point O. By D'Alembert principle the condition of equilibrium of the system

$$\sum_i \left(\vec{F_i}^{(a)} - \dot{\vec{p_i}}\right)\cdot \delta\vec{r_i} = 0$$

[redacted]

where $\delta\vec{r_i}$ = virtual displacement

$\vec{\dot{p_i}}$ = reverse effective force

$\vec{F_i}^{(a)}$ = applied external force

So $$(\vec{F_1}^{(a)} - \vec{\dot{p_1}})\cdot\delta\vec{r_1} + (\vec{F_2}^{(a)} - \vec{\dot{p_2}})\cdot\delta\vec{r_2} = 0 \quad \text{--- (1)}$$

$$\vec{F_1}^{(a)} = m_1 g \qquad r_1 + r_2 = \text{const.}$$

$$\vec{F_2}^{(a)} = m_2 g \qquad \text{or } \dot{r}_1 = -\dot{r}_2$$

$$\text{or } \ddot{r}_1 = -\ddot{r}_2$$

also $\delta r_1 = -\delta r_2$

So from (1) we get

$$\cancel{m_1 g \sin\theta_1}$$

$$(m_1 g - m_1\ddot{r}_1)\cdot\delta r_1 + (m_2 g - m_2\ddot{r}_2)\cdot\delta r_2 = 0$$

$$\Rightarrow m_1 g\sin\theta_1 - m_1\ddot{r}_1\delta r_1 + m_2 g\,\delta r_2\cdot\sin\theta_2 - m_2\ddot{r}_2\delta r_2 = 0$$

$$\text{or} \Rightarrow (m_1 g\sin\theta_1 - m_2 g)\,\delta r_1 - \ddot{r}_1(m_1 + m_2)\,\delta r_1 = 0$$

$$\Rightarrow \ddot{r}_1 = \frac{(m_1 g\sin\theta_1 - m_2 g)}{(m_1 + m_2)}$$

for equilibrium $\ddot{r}_1 = 0$

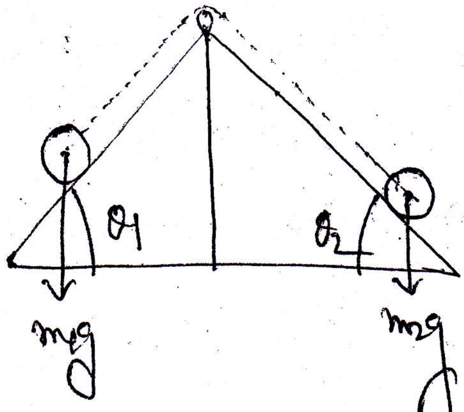
$$\therefore m_1 g\sin\theta_1 = m_2 g$$

$$\text{or } m_1\sin\theta_1 = m_2$$

$$\text{or } \boxed{\frac{m_2}{m_1} = \sin\theta_1}$$

It is the condition of equilibrium.

ii) Double inclined plane :→



In this case for equilibrium

$$(m_1 \sin\theta_1 - m_2 \sin\theta_2)g = 0$$

or
$$\frac{m_1}{m_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

or
$$\boxed{\frac{m_1}{m_2} = \frac{\sin\theta_2}{\sin\theta_1}}$$

It is the condition of equilibrium of double inclined plane.

iii) Two heavy particle of weight w_1 and w_2 are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radious R, the axis of which is horizontal. Find the condition of equilibrium of the system by applying the principle of virtual work.

Ans: According to principle of virtual work

$$\sum_{i=1}^{N} F_i \cdot \delta r_i = 0$$

Here $i = 1, 2$, and therefore

$$(W_1 \sin\theta_1 \delta r_1 + W_2 \sin\theta_2 \delta r_2) = 0$$

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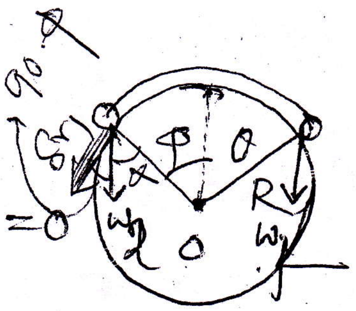
But $\delta r_1 = R d\theta$

$\delta r_2 = R d\phi$



$\therefore \quad W_1 \sin\theta \, \delta\theta + W_2 \sin\phi \, \delta\phi = 0$

But $\quad \theta + \phi = \text{const}$

$\delta\theta = -\delta\phi$

$\therefore \quad (W_1 \sin\theta - W_2 \sin\phi)\, \delta\theta = 0$

So $\Rightarrow \quad W_1 \sin\theta - W_2 \sin\phi = 0$

$$\Rightarrow \quad \boxed{\frac{W_1}{W_2} = \frac{\sin\phi}{\sin\theta}}$$

It is the condition of equilibrium.

Deduction of Lagrangean equation for a holonomic and conservative system:-

The Lagrangean equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{------} (1)$$

let us consider a system of N number of particles have f number of degree of freedom. The state of system is specified by 'f' number of generalized co-ordinate $(q_1, q_2, \cdots q_f)$. The position vector of the ith particle $\vec{r}_i$ may be expressed as a function of generalised co-ordinate and time.

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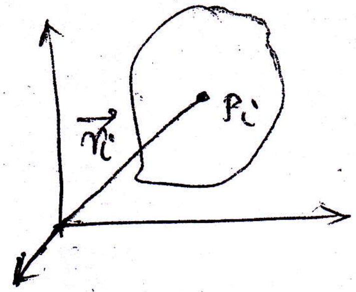
$\vec{r_i} = \vec{r_i}(q_1, q_2, \cdots q_f, t)$

Where, $i = 1, 2, 3 \cdots N$



The velocity of the ith particle $\vec{v_i} = \frac{d\vec{r_i}}{dt}$

$$\Rightarrow \vec{v_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \dot{q_j} + \frac{\partial \vec{r_i}}{\partial t} \quad \text{—} (2)$$

The virtual displacement of the ith particle

$$\delta \vec{r_i} = \sum_{ij} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j + 0 \quad \text{—} (3)$$

there is no time variation because Virtual displacement is the displacement of the coordinate only.

$\delta t = 0$

The virtual work done by the external force $\vec{F_i}$ due to virtual displacement $\delta \vec{r_i}$

$$\delta W = \sum_{i}^{N} \vec{F_i} \cdot \delta \vec{r_i}$$

$$= \sum_{ij} \vec{F_i} \cdot \left(\frac{\partial \vec{r_i}}{\partial q_j}\right) \delta q_j$$

$$\Rightarrow \delta W = \sum_{j}^{f} Q_j * \delta q_j \quad \text{—} (4)$$

Where, $Q_j = \sum_{i} \left(\vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j}\right)$, known as Component of generalized force. It may or may not have the dimension of length

From D' Alembert Principle the condition of equilibrium for the system of N particles is

$$\sum (\vec{F_i} - m_i \ddot{\vec{r_i}}) \cdot \delta \vec{r_i} = 0 \quad \text{--- (5)}$$

Then 2nd term of eqn (5)

$$\sum_i m_i \ddot{\vec{r_i}} \cdot \delta \vec{r_i}$$

$$= \sum_{ij} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

$$= \sum_{ij} \delta q_j \left[\frac{d}{dt}\left(m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right) - m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \right] \quad \text{--- (6)}$$

Differentiating partially eqn (2) with respect to $\dot{q}_j$ we get

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left[\sum \left(\frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right) = \frac{\partial r_i}{\partial q_j} \right.$$

* For holonomic const. $\frac{\partial r_i}{\partial \dot{q}_j} = 0$ = independent of velocity.

$$\cancel{= \sum \frac{\partial r_i}{\partial q_j} \delta_{ji}} = \frac{\partial r_i}{\partial q_j} \quad \text{--- (7)}$$

~~As the constraint are holonomic and~~ $\cancel{\partial \dot{q}_j}$

$$\frac{\partial v_i}{\partial q_j} = \frac{\partial^2 r_i}{\partial q_j \partial q_1} \dot{q}_1 + \frac{\partial^2 r_i}{\partial q_j \partial q_2} \dot{q}_2 + \cdots + \frac{\partial^2 r_i}{\partial q_j \partial t} \quad \text{--- (8)}$$

[redacted]

In ~~to~~ ~~anag~~ analogy to eqn (2)

$$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r_i}}{dt}\right) = \frac{\partial \vec{v_i}}{\partial \dot{q_j}} \quad \text{— (9)}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \frac{\partial \vec{v_i}}{\partial \dot{q_j}} \quad \text{— (10)}$$

Hence eqn (6) becomes

$$\sum m_i \vec{\ddot{r_i}} \cdot \delta \vec{r_i}$$

$$= \sum \delta q_j \left[\frac{d}{dt}\left(m_i \vec{v_i} \cdot \frac{\partial \vec{v_i}}{\partial \dot{q_j}}\right) - m_i \vec{v_i}\left(\frac{\partial \vec{v_i}}{\partial q_j}\right)\right]$$

[using relation (7) and (10)] — (11)

Now eqn (5) gives

$$\sum \delta q_j \left[Q_j - \frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q_j}}\left(\sum \frac{1}{2} m_i v_i^2\right)\right\} + \frac{\partial}{\partial q_j} \sum \frac{1}{2} m_i v_i^2 \right] = 0 \quad \text{— (12)}$$

The total K.E. of the system

$$T = \sum \frac{1}{2} m_i v_i^2$$

$$\therefore \sum \delta q_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_j}}\right) + \frac{\partial T}{\partial q_j}\right] = 0 \quad \text{— (13)}$$

As q_j is independent and δq_j is arbitary, we must have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_j}}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{— (14)}$$

This is the general form of lagrangian equation, true for both conservative and ~~a~~ non-conservative ~~s~~ system.
If the ~~s~~ system is conservative then $\vec{F_i}$ is $-$ve gradient of some scalar potential V, known as potential energy of the system, i.e.

$$\vec{F_i} = -\vec{\nabla_i} V = -\hat{i}\frac{\partial V}{\partial x_i} - \hat{j}\frac{\partial V}{\partial y_j} - \hat{k}\frac{\partial V}{\partial z_i}$$

The generalised component of force for conservative system becomes

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}$$

$$= -\sum \vec{\nabla_i} V \cdot \frac{\partial r_i}{\partial q_j}$$

$$= -\sum \left\{ \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_j} \right\}$$

$$\cancel{= -\sum \frac{\partial V}{\partial q_j} + \frac{\partial V}{}}$$

$$= -\frac{\partial V}{\partial q_j}(x_i, y_i, z_i)$$

$$= -\frac{dV}{dq_j}$$

Hence from eqn (14)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

[redacted]

or $\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V) = 0$

As V is not function of $\dot{q}_j$, but

$V = V(q_j)$

The quality (T−V) is called Lagrangian of the system.

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{--- (15)}$$

This is called Lagrangian equation for conservative holonomic system. It is the equation of motion is classical mechanics. It is also equivalent to Newton's equation of motion but it's form is different.

As the number of generalized coordinate is f', there are f' number of Lagrangian equation.

Newton's equation of motion from Lagrangian equation:-

The general form of the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{--- (1)}$$

Here $q_1 = x$, $q_2 = y$, $q_3 = z$ and generalised force compents are

$Q_1 = \vec{F_x}$, $Q_2 = \vec{F_y}$, $Q_3 = \vec{F_z}$

The Kinetic Energy T is

$$T = \frac{1}{2} m [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]$$

For x co-ordinate eq ① takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \quad \text{——} ②$$

But $\frac{\partial T}{\partial x} = 0$, and $\frac{\partial T}{\partial \dot{x}} = m\dot{x}$ //

Substituting in ② we get

$$\frac{d}{dt}(m\dot{x}) = F_x \quad \text{or} \quad F_x = \frac{dp_x}{dt}$$

where $p_x = m\dot{x}$ is the component of linear momentum. Similarly we can obtain

$$F_y = \frac{dp_y}{dt}, \quad \text{and} \quad F_z = \frac{dp_z}{dt}$$

Thus we get

$$\vec{F} = \frac{d\vec{p}}{dt}$$

which is Newton's equation of motion.

Cylic Co-ordinate :-

The lagrangian L of a system is a function of generalised co-ordinate q_j and generalised velocity $\dot{q}_j$ and may contain time i.e.

$$L = L(q_j, \dot{q}_j, t)$$

Where $j = 1, 2, \ldots f$.

Sometimes in physical problem it may occur that the lagrangian is not function of a particular generalised coordinate i.e

$$L \neq L(q_j)$$

Such coordinate is called the cylic coordinate. So for a cyclic coordinate q_k

$$\frac{\partial L}{\partial q_k} = 0$$

From Lagrangian equation with respect to cyclic coordinate q_k we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \underbrace{\frac{\partial L}{\partial q_k}}_{0} = 0$$

or $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

or $$\frac{\partial L}{\partial \dot{q}_k} = \text{a conserved quantity or a constant of motion}$$

$$= p_k \text{ known as}$$

generalized momentum.

So if the Lagrangian L of a system is not function of q_k i.e. q_k is cyclic, then the corresponding generalized momentum or conjugate momentum (p_k) is conserved quantity. It is the consequence of translational symmetry of Lagrangian.