# 1. Sets

## 1.1. Basic concepts.

The theory of sets in its intuitive form was developed by G. Cantor, a German mathematician, in the last part of the nineteenth century. Logicians analysed the theory in later years and built the theory on an axiomatic foundation. Here we shall introduce the naive theory of sets as was developed by Cantor. According to him 'A set is a well-defined collection of distinct objects of our perception or of our thought, to be conceived as a whole'.

Commonly we shall use capital letters $A, B, C, \ldots$ to denote sets and small letters $a, b, c, \ldots$ to denote objects (or elements) of a set. We summarise briefly some of the features of a set given in the definition.

(i) A set $S$ is a *collection of objects* (or elements) which is to be regarded as a single entity.

(ii) A set $S$ is comprised of *distinct objects* (elements) and if $a$ be an object of $S$, we denote this by $a \in S$ (read as – '$a$ belongs to $S$').

(iii) A set is *well defined*, meaning that if $S$ be a set and $a$ be an object, then either $a$ is definitely in $S$ ($a \in S$) or $a$ is definitely not in $S$, denoted by $a \notin S$ ( read as – '$a$ does not belong to $S$').

A set may be described by the characterising property of its elements. If $p$ be the defining property of the elements of $S$ then $S$ is expressed as $S = \{x : x \text{ has the property } p\}$. The brace notation for $S$ expresses the fact that $S$ is the collection of all those elements $x$ such that $x$ has the property $p$.

**Examples.**

1. The set $S$ of all integers is expressed as $S = \{x : x \text{ is an integer}\}$. Here $3 \in S$ but $\frac{1}{3} \notin S$.

2. The set $T$ of all prime numbers less than 20 is expressed as
$T = \{x : x \text{ is a prime less than } 20\}$. Here $3 \in T$ but $4 \notin T$.

A set may also be described by listing all elements of the collection. Thus the set in Example 2 can also be described as

$$T = \{2, 3, 5, 7, 11, 13, 17, 19\}.$$

Throughout this text we shall use some accepted notations for the familiar sets of numbers.

| | |
|---|---|
| $\mathbb{N}$ | is the set of all natural numbers |
| $\mathbb{Z}$ | is the set of all integers |
| $\mathbb{Z}^+$ | is the set of all positive integers |
| $\mathbb{Q}$ | is the set of all rational numbers |
| $\mathbb{Q}^+$ | is the set of all positive rational numbers |
| $\mathbb{R}$ | is the set of all real numbers |
| $\mathbb{R}^+$ | is the set of all positive real numbers |
| $\mathbb{C}$ | is the set of all complex numbers. |

## 1.2. Subset.

Let $S$ be a set. A set $T$ is said to be a *subset* of $S$ if $x \in T \Rightarrow x \in S$. This means that each element of $T$ is an element of $S$. This is expressed symbolically by $T \subset S$ signifying that $T$ is contained in $S$.

If $T$ be a subset of $S$, $S$ is said to be a *superset* of $T$ and this is expressed by $S \supset T$.

For example, $\mathbb{Z} \subset \mathbb{Q}$, $\mathbb{Q} \subset \mathbb{R}$.

We conceive of the existence of a set containing no element. This is called the *null set*, or the *empty set*, or the *void set* and is denoted by $\phi$. For logical consistency, the null set is taken to be a subset of every set.

Therefore for every set $S$, $S \subset S$ and $\phi \subset S$. $S$ and $\phi$ are said to be the *improper subsets* of $S$. Any other subset of $S$ is said to be a *proper subset* of $S$.

**Definition.** A set is said to be a *finite set* if either it is empty or it contains a finite number of elements; otherwise it is said to be an *infinite set*.

**Definition.** Two sets $S$ and $T$ are said to be *equal* (expressed by $S = T$) if $S$ is a subset of $T$ and $T$ is a subset of $S$. Two equal sets contain precisely the same elements.

The relation $\subset$ is called *set inclusion*. The main properties of the inclusion relation are the following:

(i) $S \subset S$ for every set $S$ (reflexivity)
(ii) $S \subset T$ and $T \subset S \Rightarrow S = T$ (antisymmetry)
(iii) $S \subset T$ and $T \subset V \Rightarrow S \subset V$ (transitivity).

There may, however, exist sets $S$ and $T$ such that neither $S \subset T$ nor $T \subset S$. Such a pair of sets is said to be *incomparable*.

For example, let $S$ be the set of all integral multiples of 2, $T$ be the set of all integral multiples of 3. Then neither $S \subset T$ nor $T \subset S$.



**Examples.**

**1.** The set of all natural numbers less than 20 is a subset of the set $\mathbb{N}$. It is denoted by $\{n \in \mathbb{N} : n < 20\}$.

**2.** The set of all integers greater than 5 is a subset of the set $\mathbb{Z}$. It is denoted by $\{x \in \mathbb{Z} : x > 5\}$.

**3.** The set of all positive rational numbers is a subset of the set $\mathbb{Q}$. It is denoted by $\{x \in \mathbb{Q} : x > 0\}$.

**4.** The set of all real numbers lying between 1 and 2 is a subset of the set $\mathbb{R}$. It is denoted by $\{x \in \mathbb{R} : 1 < x < 2\}$.

**5.** The set of all complex numbers of unit modulus is a subset of the set $\mathbb{C}$. It is denoted by $\{z \in \mathbb{C} : |z| = 1\}$.

## 1.3. Algebraic operations on sets.

In this section we shall discuss several ways of combining different sets and develop some properties among them. For this purpose we shall consider several sets, in a particular discussion, as subsets of a single fixed set, called the *universal set* in relation to its subsets. A universal set is generally denoted by $U$.

Let $U$ be the universal set and $A, B, C, \cdots$ be the subsets of $U$. We define the following operations on the class of subsets of $U$.

**(a) Union.** The *union* (or *join*) of two subsets $A$ and $B$ is a subset of $U$, denoted by $A \cup B$ and is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Therefore $A \cup B$ is the set of all those elements which belong either to $A$, or to $B$, or to both. It follows that $A \subset A \cup B, B \subset A \cup B$.

For example, let $U = \{1, 2, 3, \ldots, 10\}$ be the universal set and $A = \{1, 3, 5, 7\}$, $B = \{5, 7, 9\}$. Then $A \cup B = \{1, 3, 5, 7, 9\}$.

**(b) Intersection.** The *intersection* (or *meet*) of two subsets $A$ and $B$ is a subset of $U$, denoted by $A \cap B$ and is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Therefore $A \cap B$ is the set of all those elements which belong to both $A$ and $B$. It follows that $A \cap B \subset A$, $A \cap B \subset B$.

For example, let $U = \{1, 2, 3, \ldots, 10\}$ be the universal set and $A = \{1, 3, 5, 7\}$, $B = \{5, 7, 9\}$. Then $A \cap B = \{5, 7\}$.

**Definition.** If two subsets $A$ and $B$ have no common element then $A \cap B = \phi$ and $A$ and $B$ are said to be *disjoint*.

**Properties of union and intersection.**

**1.** *Consistency property.* The three relations $B \subset A$, $A \cup B = A$ and $A \cap B = B$ are mutually equivalent, i.e., one implies the other two.

The following properties can be easily deduced from the consistency property.

**1a.** $A \cup \phi = A$, $A \cap \phi = \phi$. This follows from the relation $\phi \subset A$.

**1b.** $A \cup U = U$, $A \cap U = A$. This follows from the relation $A \subset U$.

**1c.** $A \cup A = A$, $A \cap A = A$. (*Idempotent property*).
This follows from $A \subset A$.

**1d.** $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$. (*Absorptive property*). This follows from $A \cap B \subset A \subset A \cup B$.

**2.** *Commutative property.* $A \cup B = B \cup A, A \cap B = B \cap A$.

**3.** *Associative property.*
$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

**4.** *Distributive properties.*
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proofs of 2 and 3 are left as exercise.

*Proof* of 4. Let $P = A \cup (B \cap C), Q = (A \cup B) \cap (A \cup C)$.

Let $x$ be an element of $P$.

$x \in P \Rightarrow x \in A$ or $x \in B \cap C$
$\Rightarrow x \in A$ or $\{x \in B$ and $x \in C\}$
$\Rightarrow \{x \in A$ or $x \in B\}$ and $\{x \in A$ or $x \in C\}$
$\Rightarrow x \in A \cup B$ and $x \in A \cup C$
$\Rightarrow x \in (A \cup B) \cap (A \cup C)$.

Therefore $P \subset Q$ ... (i)

Let $y$ be an element of $Q$.

$y \in Q \Rightarrow y \in A \cup B$ and $A \cup C$
$\Rightarrow \{y \in A$ or $y \in B\}$ and $\{y \in A$ or $y \in C\}$
$\Rightarrow y \in A$ or $\{y \in B$ and $y \in C\}$
$\Rightarrow y \in A$ or $y \in B \cap C$
$\Rightarrow y \in A \cup (B \cap C)$.

Therefore $Q \subset P$ ... (ii)

Combining (i) and (ii), we have $P = Q$.



Let $M = A \cap (B \cup C)$, $N = (A \cap B) \cup (A \cap C)$.

Let $x$ be an element of $M$.

$x \in M \Rightarrow x \in A$ and $x \in B \cup C$
$\Rightarrow x \in A$ and $\{x \in B$ or $x \in C\}$
$\Rightarrow \{x \in A$ and $x \in B\}$ or $\{x \in A$ and $x \in C\}$
$\Rightarrow x \in A \cap B$ or $x \in A \cap C$
$\Rightarrow x \in (A \cap B) \cup (A \cap C)$.

Therefore $M \subset N$ ... (iii)

Let $y$ be an element of $N$.

$y \in N \Rightarrow y \in A \cap B$ or $A \cap C$
$\Rightarrow \{y \in A$ and $y \in B\}$ or $\{y \in A$ and $y \in C\}$
$\Rightarrow y \in A$ and $\{y \in B$ or $y \in C\}$
$\Rightarrow y \in A \cap (B \cup C)$.

Therefore $N \subset M$ ... (iv)

Combining (iii) and (iv), we have $M = N$.

**(c) Complementation.** The *complement* of a subset $A$ is a subset of $U$, denoted by $A'$ (or $A^c$) and is defined by

$$A' = \{x \in U : x \notin A\}.$$

$A'$ consists of all those elements of $U$ which are not in $A$.

For example, let $U = \{1, 2, 3, 4, 5, 6\}$ be the universal set and $A = \{1, 3, 5\}$. Then $A' = \{2, 4, 6\}$.

**Properties of complementation.**

**5.** $A \cup A' = U$, $A \cap A' = \phi$ for any subset $A$.

**6.** $(A')' = A$ for any subset $A$.

Proofs of 5 and 6 are left as exercise.

**7. De Morgan's laws. 7a.** $(A \cup B)' = A' \cap B'$.
**7b.** $(A \cap B)' = A' \cup B'$.

*Proof.* **7a.** Let $P = (A \cup B)'$, $Q = A' \cap B'$

Let $x$ be an element of $P$. $x \in P \Rightarrow x \notin A \cup B$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$.

Therefore $P \subset Q$ ... (i)

Let $y$ be an element of $Q$. $y \in Q \Rightarrow y \in A'$ and $y \in B'$
$\Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow y \notin A \cup B$
$\Rightarrow y \in (A \cup B)'$.

Therefore $Q \subset P$ ... (ii)

Combining (i) and (ii), we have $P = Q$.

*Proof.* **7b.** Let us consider two subsets $A'$ and $B'$.

By 7a, $(A' \cup B')' = (A')' \cap (B')'$
or, $(A' \cup B')' = A \cap B$, by property 6
or, $\{(A' \cup B')'\}' = (A \cap B)'$, taking complements
or, $A' \cup B' = (A \cap B)'$, by property 6.

The independent proof of 7b is left as exercise.

**(d) Difference.** The *difference* of two subsets $A$ and $B$ is a subset of $U$, denoted by $A - B$ and is defined by

$$A - B = \{x \in A : x \notin B\}.$$

$A - B$ is a subset of $A$ and is the set of all those elements of $A$ which are not in $B$.

In particular, $A - B = \phi$ if $A \subset B$ and $A - B = A$ if $A \cap B = \phi$.

The subset $A - B$ is also called the *complement of B relative to A*.

The difference $A - B$ can be expressed in terms of the complement as $A - B = A \cap B'$.

**Properties of difference.**

**8a.** $A - (B \cap C) = (A - B) \cup (A - C)$.

**8b.** $A - (B \cup C) = (A - B) \cap (A - C)$.

*Proof.* 8a. $A - (B \cap C) = A \cap (B \cap C)'$
$= A \cap (B' \cup C')$
$= (A \cap B') \cup (A \cap C')$
$= (A - B) \cup (A - C)$.

Proof of 8b is left as exercise.

$A - (B \cup C)$ is the *relative complement* of $B \cup C$ in $A$ and $A - (B \cap C)$ is the *relative complement* of $B \cap C$ in $A$.

These properties are the generalisation of De Morgan's Laws.



**(e) Symmetric Difference.** The *symmetric difference* of two subsets $A$ and $B$ is a subset of $U$, denoted by $A \Delta B$ and is defined by

$$A \Delta B = (A - B) \cup (B - A).$$

$A \Delta B$ is the set of all those elements which belong either to $A$ or to $B$ but not to both.

$A \Delta B$ is also expressed as $(A \cup B) - (A \cap B)$.

$A \Delta B$ can also be expressed as $(A \cap B') \cup (A' \cap B)$.

It follows from the definition that $A \Delta \phi = A$ for all subsets $A$ and $A \Delta A = \phi$ for all subsets A.

**Note.** $(A - B) \cap (B - A) =$
$(A \cap B') \cap (B \cap A') = A \cap (B' \cap B) \cap A' = (A \cap \phi) \cap A' = \phi \cap A' = \phi$.

Therefore $A \Delta B$ can be considered as the union of two *disjoint* subsets $A - B$ and $B - A$, provided $A - B$ and $B - A$ are both non-empty.

**Properties of symmetric difference.**

**9.** $A \Delta B = B \Delta A$. (*Commutative property*).

**10.** $A \Delta (B \Delta C) = (A \Delta B) \Delta C$. (*Associative property* ).

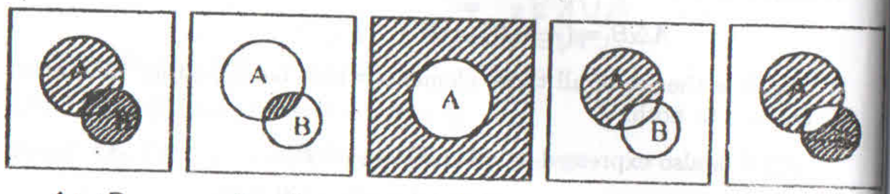
*Proof.* $B \Delta C = (B - C) \cup (C - B)$
$= (B \cap C') \cup (B' \cap C)$.

$A \Delta (B \Delta C)$
$= \{A \cap [(B \cap C') \cup (B' \cap C)]'\} \cup \{A' \cap [(B \cap C') \cup (B' \cap C)]\}$
$= \{A \cap [(B' \cup C) \cap (B \cup C')]\} \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$
$= \{A \cap [(B' \cap C') \cup (C \cap B)]\} \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$
$= (A \cap B' \cap C') \cup (A \cap B \cap C) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$
$= (A \cap B \cap C) \cup (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$.

The right hand expression remains same if $A$ and $C$ are interchanged.

Therefore $A \Delta (B \Delta C) = C \Delta (B \Delta A)$
$= (B \Delta A) \Delta C$, by 9
$= (A \Delta B) \Delta C$, by 9.

The set operations – union, intersection, complementation, difference and symmetric difference can be visualised from the diagrammatic representation of sets, called the *Venn diagram.*




A ∪ B    A ∩ B    A′    A − B    A Δ B

The rectangular region represents the universal set $U$ and the circular regions the subsets $A$ and $B$. The shaded portion represents the set named below the diagram.

The properties of different set operations can be verified from the Venn diagram.

**Worked Examples.**

**1.** If $A$, $B$, $C$ be subsets of a universal set $U$ prove that

$$(A\cup B)\cap(B\cup C)\cap(C\cup A)=(A\cap B)\cup(B\cap C)\cup(C\cap A).$$

$$\begin{aligned}(A\cup B)\cap(B\cup C) &= (B\cup A)\cap(B\cup C)\\ &= B\cup(A\cap C) \quad [\text{ distributive law }]\\ &= B\cup(C\cap A).\end{aligned}$$

$$\begin{aligned}\text{L.H.S.} &= (A\cup B)\cap(B\cup C)\cap(C\cup A)\\ &= [B\cup(C\cap A)]\cap(C\cup A)\\ &= [B\cap(C\cup A)]\cup[(C\cap A)\cap(C\cup A)] \quad [\text{distributive law}]\\ &= [(B\cap C)\cup(B\cap A)]\cup(C\cap A) \quad [C\cap A\subset C\cup A]\\ &= (A\cap B)\cup(B\cap C)\cup(C\cap A)=\text{R.H.S.(proved).}\end{aligned}$$

**2.** Prove that $A\cap(B\Delta C)=(A\cap B)\Delta(A\cap C)$.

$$\begin{aligned}\text{R.H.S.} &= (A\cap B)\Delta(A\cap C)\\ &= [(A\cap B)\cap(A\cap C)']\cup[(A\cap B)'\cap(A\cap C)]\\ &= [(A\cap B)\cap(A'\cup C')]\cup[(A'\cup B')\cap(A\cap C)]\\ &= [(A\cap B\cap A')\cup(A\cap B\cap C')]\cup[(A'\cap A\cap C)\cup(B'\cap A\cap C)]\\ &= (A\cap B\cap C')\cup(A\cap B'\cap C) \quad [A\cap B\cap A'=\phi, A'\cap A\cap C=\phi]\\ &= A\cap[(B\cap C')\cup(B'\cap C)]\\ &= A\cap(B\Delta C)=\text{ L.H.S. (proved).}\end{aligned}$$



## 1.4. Set of sets.

We have defined a set as a collection of its elements. If the elements be sets themselves, then we have a *family of sets*, or a *set of sets*.

For example, the collection of all subsets of a non-empty set $S$ is a set of sets. This set is said to be the *power set* of $S$ and is denoted by $P(S)$.

If $S$ contains $n$ elements, then $P(S)$ contains $2^n$ subsets, because a subset of $P(S)$ is either $\phi$ or a subset containing $r$ elements of $S, r = 1,2,\ldots,n$.

Let $I$ be the finite set $\{1,2,\ldots,n\}$ and $\mathcal{F}$ be the family of $n$ sets $A_1, A_2, \ldots, A_n$. $\mathcal{F}$ is expressed as

$$\mathcal{F}=\{A_\alpha:\alpha\in I\}.$$

$I$ is called the *index set* of the family $\mathcal{F}$. The members of $\mathcal{F}$ are being indexed by $I$.

Let $I$ be the set $\mathbb{N}$. Then the family of sets $\mathcal{F}=\{A_\alpha:\alpha\in I\}$ is an infinite collection of the sets $A_\alpha$ where each $A_\alpha$ corresponds to a natural number. The family is also equivalently expressed as

$$\mathcal{F}=\{A_n:n\in\mathbb{N}\}.$$

Let $I$ be an arbitary set. Then the family of sets $\mathcal{F}=\{A_\alpha:\alpha\in I\}$ is an arbitrary collection of sets $A_\alpha$ indexed by $I$.

Let $I$ be an index set and $\{A_\alpha:\alpha\in I\}$ be a family of subsets of a universal set $S$.

Then the union of the family of these subsets is defined by

$$\bigcup_{\alpha\in I}A_\alpha=\{x\in S:x\in A_\alpha \text{ for at least one } \alpha \text{ in } I\}.$$

The intersection of the family of these subsets is defined by

$$\bigcap_{\alpha\in I}A_\alpha=\{x\in S:x\in A_\alpha \text{ for all } \alpha\in I\}.$$

In particular, for a finite family of subsets $\mathcal{F}=\{A_1,A_2,\ldots,A_n\}$

(i) $\bigcup_{i=1}^{n}A_i=\{x:x\in A_i \text{ for at least one } i\}$;

(ii) $\bigcap_{i=1}^{n}A_i=\{x:x\in A_i \text{ for each } i\}$;

(iii) $\left(\bigcup_{i=1}^{n}A_i\right)'=\bigcap_{i=1}^{n}A_i'$;

(iv) $\left(\bigcap_{i=1}^{n}A_i\right)'=\bigcup_{i=1}^{n}A_i'$.

**Examples.**

1. Let $S = \{1,2,3\}$. Then the power set of $S$ is given by
$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, S\}$.

2. For each $n \in \mathbb{N}$, let $I_n = [0, 1/n]$. Then $I_1 \supset I_2 \supset I_3 \supset \cdots\cdots$ and $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$.

3. For each $n \in \mathbb{N}$, let $I_n = (0, 1/n)$. Then $I_1 \supset I_2 \supset I_3 \supset \cdots\cdots$ and $\bigcap_{n\in\mathbb{N}} I_n = \phi$.

## 1.5. Partition of a set.

Let $S$ be a non-empty set. A family of non-empty subsets $\{S_\alpha : \alpha \in I\}$, $I$ being the index set, is said to form a *partition* of $S$ if

(i) $\bigcup_{\alpha\in I} S_\alpha = S$

and (ii) $S_\alpha \cap S_\beta = \phi$ for $\alpha, \beta \in I$ and $\alpha \neq \beta$.

The condition (ii) says that the family of subsets $\{S_\alpha : \alpha \in I\}$ is pairwise disjoint.

In particular, a finite family of non-empty subsets $S_1, S_2, \ldots, S_n$ of a non-empty set $S$ is said to form a partition of $S$ if

(i) $S_1 \cup S_2 \cup \cdots \cup S_n = S$

and (ii) $S_i \cap S_j = \phi$ for $i \neq j$.

**Examples.**

1. Let $A$ and $B$ be non-empty sets such that the sets $A - B$, $B - A$ and $A \cap B$ are each non-empty. Then the sets $A - B$, $B - A$ and $A \cap B$ form a partition of the set $A \cup B$, because

(i) $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$, and

(ii) $A - B$, $B - A$ and $A \cap B$ are mutually disjoint subsets of $A \cup B$.

2. Let $S$ be a non-empty set and $A$ be a proper subset of $S$. Then the two subsets $A$ and $A'$ are such that each is non-empty and

(i) $A \cup A' = S$, (ii) $A \cap A' = \phi$.

Therefore the subsets $A$ and $A'$ form a partition of $S$.

3. Let $S$ be a non-empty set and $A, B$ are non-empty subsets of $S$ such that each of $A \cap B, A \cap B', A' \cap B, A' \cap B'$ is non-empty.

Let $M_1 = A \cap B, M_2 = A \cap B', M_3 = A' \cap B, M_4 = A' \cap B'$.

Then $M_1 \cup M_2 \cup M_3 \cup M_4 = S$ and $M_i \cap M_j = \phi, i \neq j$.

Therefore the subsets $M_1, M_2, M_3, M_4$ form a partition of $S$.



4. Let $C_0, C_1, C_2, C_3$ be four subsets of $\mathbb{Z}$ defined by

$$\begin{aligned}
C_0 &= \{4n : n \text{ is an integer}\} &&= \{0, \pm 4, \pm 8, \cdots\},\\
C_1 &= \{4n+1 : n \text{ is an integer}\} &&= \{1, 1\pm 4, 1\pm 8, \cdots\},\\
C_2 &= \{4n+2 : n \text{ is an integer}\} &&= \{2, 2\pm 4, 2\pm 8, \cdots\},\\
C_3 &= \{4n+3 : n \text{ is an integer}\} &&= \{3, 3\pm 4, 3\pm 8, \cdots\}.
\end{aligned}$$

Then the subsets $C_0, C_1, C_2, C_3$ are such that each is non-empty and

$C_0 \cup C_1 \cup C_2 \cup C_3 = \mathbb{Z}$ and $C_i \cap C_j = \phi, i \neq j$.

Therefore the subsets $C_0, C_1, C_2, C_3$ form a partition of the set $\mathbb{Z}$.

## 1.6. Cartesian product of sets.

Let $A$ and $B$ be non-empty sets. The *Cartesian product* of $A$ and $B$, denoted by $A \times B$, is the set defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

$A \times B$ is the set of all ordered pairs $(a, b)$, the first component being an element of $A$ and the second being an element of $B$.

Let $A_1, A_2, \ldots, A_n$ be a finite collection of non-empty sets. The Cartesian product of the collection, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) : x_i \in A_i\}.$$

In particular, if $A_1 = A_2 = \cdots = A_n$, the Cartesian product of the collection of sets, denoted by $A^n$, is the set of all ordered $n$-tuples $\{(x_1, x_2, \ldots, x_n) : x_i \in A\}$.

**Examples.**

1. Let $A = \{1, 3, 5\}, B = \{2, 4\}$.
Then $A \times B = \{(1,2), (1,4), (3,2), (3,4), (5,2), (5,4)\}$;
$B \times A = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5)\}$.

2. $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs $\{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

3. $\mathbb{R} \times \mathbb{R}$( or $\mathbb{R}^2$) is the set of all ordered pairs$\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$.

4. $\mathbb{R}^3$ is the set of all ordered triplets $\{(x_1, x_2, x_3) : x_i \in \mathbb{R}\}$.

5. $\mathbb{R}^n$ is the set of all ordered $n$-tuples $\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.

6. Let $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. Then $A \times B = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 1 \leq y \leq 2\}$. $A \times B$ is the square region in $\mathbb{R}^2$ (Cartesian plane) bounded by the lines $x = 0, x = 1, y = 1, y = 2$.