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PROGRAMME: B.Sc. PHYSICS HONOURS

SEMESTER - VI

Course: Electromagnetic Theory (CC-XIII)

Lecture Note: 2

Conservation laws in Electrodynamics

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Conservation Laws

8.1 Charge and Energy

8.1.1 The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics. But I want to begin by reviewing the conservation of *charge*, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure—that's **global** conservation of charge; but **local** conservation of charge is a much stronger statement: If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface. The tiger can't simply rematerialize outside the cage; if it got from inside to outside it must have found a hole in the fence.

Formally, the charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau, \tag{8.1}$$

and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$, so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{8.2}$$

Using Eq. 8.1 to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau, \tag{8.3}$$

and since this is true for *any* volume, it follows that

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.} \tag{8.4}$$

This is, of course, the continuity equation—the precise mathematical statement of local conservation of charge. As I indicated earlier, it can be derived from Maxwell's equations—conservation of charge is not an *independent* assumption, but a *consequence* of the laws of electrodynamics.

The purpose of this chapter is to construct the corresponding equations for conservation of energy and conservation of momentum. In the process (and perhaps more important) we will learn how to express the energy density and the momentum density (the analogs to ρ), as well as the energy "current" and the momentum "current" (analogous to $\mathbf{J}$).

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Electromagnetic wave :-

The four Maxwell's eqns provide us all information that ~~can~~ can be drawn from the classical theory of electric and magnetic field. The most important of these is that the field generates by moving charges ~~an~~ and current are found to leave the source and travell in space in the form of e.m. wave. Its need no medium for its propagation. The velocity of em wave depends upon the permittivity (ϵ) and permeability (μ) of the medium.

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The Maxwell equation in matter with no free charge, no conductivity are

i) $\vec{\nabla}\cdot\vec{E} = 0$ ——— (1)

ii) $\vec{\nabla}\cdot\vec{B} = 0$ ——— (2)

iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ——— (3)

iv) $\nabla\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$ ——— (4)

Taking Curl on both sides of eq^n (iii)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$= -\frac{\partial}{\partial t}\left(\mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

or $$\nabla(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

or $$\vec{\nabla}^2\vec{E} = \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

or $$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\mu\epsilon}\nabla^2\vec{E} = 0 \quad \text{——— (5)}$$

or $$\frac{\partial^2 \vec{E}}{\partial t^2} - v^2\nabla^2\vec{E} = 0$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, velocity of wave in the medium.

Taking Curl on both sides of eq^n (iv)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu\epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or $$\nabla(\nabla\cdot\vec{B}) - \vec{\nabla}^2\vec{B} = \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

or $$\nabla^2\vec{B} = \mu\epsilon\frac{\partial^2 B}{\partial t^2}$$

or $$\frac{\partial^2 \vec{B}}{\partial t^2} - \frac{1}{\mu\epsilon}\nabla^2\vec{B} = 0$$

or $$\frac{\partial^2 \vec{B}}{\partial t^2} - v^2\nabla^2\vec{B} = 0 \quad \text{——— (6)}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, velocity of wave.

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The equation ⑤ and eqn ⑥ are the three-dimensional wave equation. So the dynamic electric and magnetic field produced by varying charge or current, does not remain in a limited space but travels three-dimensionally as electro-magnetic wave with velocity

$v = \frac{1}{\sqrt{\mu\epsilon}}$. In vacuum $v \to c$

and $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8$ m/s

So the two constant, ϵ_0 in ~~Cou~~ Coulomb law and μ_0 in Bio-Savart law, are related to the speed of light in a simple manner.

The refractive index ~~of~~ of a medium

$$n = \frac{c}{v}$$

$$= \sqrt{\frac{\mu\epsilon}{\mu_0 \epsilon_0}} = \sqrt{\mu_r \epsilon_r}.$$

where ϵ_r, μ_r are relative one.

For most of the transparent medium $\mu_r \simeq 1$. So then the refractive index becomes

$$n = \sqrt{\epsilon_r} \text{ (approximately)}.$$

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The simple types of solution of eqn (5) and (5) is the plane wave solution given by

$$\vec{E}(x,y,z,t) = \hat{\epsilon}_1 E_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}.$$

$$\vec{B}(r,t) = \hat{\epsilon}_2 B_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}$$

where E_0 and B_0 are the complex amplitude. $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are two unit vector representing the direction of $\vec{E}$ and $\vec{B}$. They are called Polarization vectors.

ω is the frequency $= 2\pi c/\lambda$.

and $\vec{K}$ is the propagation vector.

$$|\vec{K}| = 2\pi/\lambda$$

Electromagnetic Energy & Momentum:

The em wave when passes through vaccum and medium, it carries energy and momentum in itself. (even with-out mass)

It can be shown, the energy carried by an em wave per second per unit area, is given by

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0},$$ known as pointing vector.

The momentum density carried by the em wave is

$$\vec{g}_{em} = \epsilon_0(\vec{E}\times\vec{B}) = \vec{S}/c^2 \left[c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \right]$$

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The electrostatic potential energy of the system of charges producing electric field per unit volume is given by

$$u_e = \frac{1}{2}\epsilon_0 E^2 \quad (\text{in vacuum})$$
$$= \frac{1}{2}(\vec{E}\cdot\vec{D}) \quad (\text{in medium}).$$

Similarly the magnetic energy density in a magnetic field $\vec{B}$

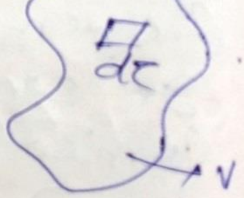
$$u_m = \frac{1}{2}\frac{B^2}{\mu_0} \quad (\text{in vacuum})$$
$$= \frac{1}{2}(\vec{B}\cdot\vec{H}) \quad (\text{in medium}).$$

So the electro-magnetic energy density of an electric field $\vec{E}$ and magnetic field $\vec{B}$ is

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + B^2/\mu_0\right) \ \mathrm{J/m^3} \quad (1)$$

The total em energy ~~density~~ in a region of volume V.

$$U_{em} = \frac{1}{2}\int_V \left(\epsilon_0 E^2 + B^2/\mu_0\right) d\tau \quad (2)$$



where $d\tau$ is volume element.

The Lorentz force acting on the charge q moving with velocity $\vec{u}$ in an em field

$$\vec{F}_{em} = q\left[\vec{E} + \vec{u}\times\vec{B}\right]$$

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The rate of doing work on this charge by the em field is

$$\frac{dw}{dt} = \vec{F}_{em}\cdot\vec{u}$$
$$= q\left[\vec{E}\cdot\vec{u} + (\vec{u}\times\vec{B})\cdot\vec{u}\right]$$
$$= q\left[\vec{E}\cdot\vec{u}\right] \quad \left[\because = 0\right]$$

For a Continuous distribution of charge in an ~~a~~ enclosure of volume V in an em field, the rate of work done

$$P = \frac{dw}{dt} = \int_V (\vec{E}\cdot\vec{u})\rho\, d\tau$$

where ρ is the volume charge density.

$$\text{or } \frac{dw}{dt} = \int_V (\vec{E}\cdot\vec{J})\, d\tau \quad \text{—— (3)}$$

where $\vec{J} = \vec{u}\rho$, Current density.

So $(\vec{E}\cdot\vec{J})$ has the significance and it is work done per sec. per unit volume containing charges, or $(\vec{E}\cdot\vec{J})$ represents the power derived per unit volume.

From Maxwell's forth equation,

$$\vec{\nabla}\times\vec{B} = \mu_0\left(J + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\text{or } \vec{J} = \left(\frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \quad \text{—— (4)}$$

$$\text{or } \vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\,E\cdot(\vec{\nabla}\times\vec{B}) - \epsilon_0\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right).$$

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Now from vector Calculus

$$\nabla\cdot(\bar{E}\times\bar{B}) = \vec{B}\cdot(\bar{\nabla}\times\bar{E}) - \bar{E}\cdot(\bar{\nabla}\times\bar{B})$$

or $$E\cdot(\nabla\times B) = \vec{B}\cdot(\bar{\nabla}\times\bar{E}) - \nabla\cdot(\bar{E}\times\bar{B})$$

So $$\bar{E}\cdot\bar{J} = \frac{1}{\mu_0}\vec{B}\cdot(\bar{\nabla}\times\bar{E}) - \frac{1}{\mu_0}\bar{\nabla}\cdot(\bar{E}\times\bar{B}) - \epsilon_0\left(\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}\right).$$

or $$\bar{E}\cdot\bar{J} = -\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\bar{B}}{\partial t} - \epsilon_0\bar{E}\cdot\frac{\partial\bar{E}}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\bar{E}\times\bar{B}). \quad (5)$$

Now hence from (3)

$$\frac{dW}{dt} = -\int_V\left(\frac{1}{\mu_0}\bar{B}\cdot\frac{\partial\bar{B}}{\partial t} + \epsilon_0\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}\right)d\tau - \int_V\nabla\cdot\left(\frac{\bar{E}\times\bar{B}}{\mu_0}\right)d\tau.$$

$$= -\frac{\partial}{\partial t}\int_V\left(\frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0E^2\right)d\tau - \int_V\nabla\cdot\left(\frac{\bar{E}\times\bar{B}}{\mu_0}\right)d\tau$$

Now using eqn (1) we get

$$\frac{dW}{dt} = -\frac{\partial U_{em}}{\partial t} - \int_V(\bar{\nabla}\cdot\bar{S})\,d\tau \quad (6)$$

where $\vec{S} = \left(\frac{\bar{E}\times\bar{B}}{\mu_0}\right)$, a vector along $\vec{E}\times\vec{B}$, called poynting vector

or $$\frac{dW}{dt} = -\frac{\partial U_{em}}{\partial t} - \oint\!\!\oint \vec{S}\cdot d\vec{A} \quad (7)$$

where $d\vec{A}$ is surface element of the surface $\bar{A}$ enclosing volume ~~D~~ V.

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The equation (7) is called the pointing theorem. The 1st term in eqn (7)

$$\frac{\partial U_{em}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\int\left(\epsilon_0 E^2 + B^2/\mu_0\right) d\tau$$

represents the rate at which total em ~~energy~~ energy decreased ~~stored~~ in the em field and the 2nd term represents the rate at which the em energy is carried out of volume v through its surface by the fields. So eqn (7) represents work energy theorem in electrodynamics and theorem is known as pointing theorem.

This theorem states that, the total workdone per sec on the charges in the volume v by em field forces is equal to rate of decrease of em energy in it less the energy which flows through its surface enclosing volume v per second.

The energy flow per second through unit area normally i.e the energy density flux vector is represented by

$$\vec{S} = \frac{(\vec{E}\times\vec{B})}{\mu_0}$$, known as pointing vector. It's unit is watt/m² and direction is along the energy propagation.

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The work done on the charges will increase their mechanical energy (K.E.+ P.E.) and if U_{mech} is the mechanical energy density then

$$\frac{dw}{dt} = \frac{d}{dt}\int_V U_{mech}\, d\tau \quad (8)$$

= total energy gain/sec

Then eqn (7) takes the form

$$\frac{d}{dt}\int_V (U_{mech} + U_{em})\, d\tau = -\oiint_S \vec{S}\cdot d\vec{a}$$

$$= -\int_V (\vec{\nabla}\cdot\vec{S})\, d\tau$$

$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(U_{mech} + U_{em}) + (\vec{\nabla}\cdot\vec{S})\right] d\tau = 0$$

As $d\tau$ is arbitary, so we get

$$\frac{\partial}{\partial t}(U_{mech} + U_{em}) = -\vec{\nabla}\cdot\vec{S}$$

or $$\boxed{\vec{\nabla}\cdot\vec{S} = -\frac{\partial}{\partial t}(U_{mech} + U_{em})} \quad (9)$$

This is the differential form of pointing theorem. It simply represents the conservation of energy. It is equation of continuity in electrodynamics.

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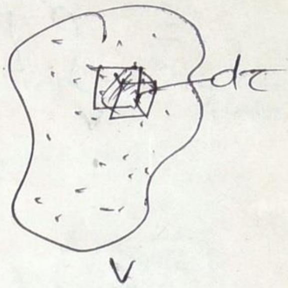
Electromagnetic momentum :-

The electromagnetic wave while travelling from one place to another carried also momentum (mass less). The Lorentz force acting by the e.m. field on all charges in a volume V is

$$\vec{F} = \int_V \vec{f}\, d\tau ,$$



where $\vec{f}$ is the force density i.e force per unit volume.

$$\vec{F} = \int_V \rho(\bar{E} + \bar{u}\times\bar{B})\, d\tau$$

$$\vec{F} = \frac{d}{dt}\int \vec{g}_{mech}\, d\tau \quad \text{--- (1)}$$

where $\vec{g}_{mech}$ is the mechanical momentum density.

The mechanical ~~momentum~~ force density

$$\vec{f} = \frac{d}{dt}\vec{g}_{mech} ,$$

Now from (1)

$$\vec{F} = \int_V (\rho\bar{E} + \bar{J}\times\bar{B})\, d\tau \quad \text{--- (2)}$$

where $\vec{J} = \rho\vec{u}$, current density.

from eq^n (1) and (2)

$$\frac{d}{dt}\int_V \vec{g}_{mech}\, d\tau = \int_V (\rho\bar{E} + \bar{J}\times\bar{B})\, d\tau$$

Using maxwell's equation

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$$= \int_V \left[\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B} - \epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\right)\times\vec{B}\right] d\tau$$

as $\rho = (\vec{\nabla}\cdot\vec{E})\epsilon_0$, and $\vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) - \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$

$$= \int_V \left[\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}\left[\vec{B}\times(\vec{\nabla}\times\vec{B})\right] - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\times\vec{B}\right] d\tau$$

$$= \int_V \left[\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\times\vec{B}\right] d\tau$$

(as $(\vec{\nabla}\cdot\vec{B}) = 0$ we can introduce this term)

Now $\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial \vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial \vec{B}}{\partial t}$

$$= \int_V \left[\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}(\vec{B}\times(\vec{\nabla}\times\vec{B})) + \epsilon_0 \vec{E}\times\frac{\partial \vec{B}}{\partial t} - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right] d\tau$$

Now

$$\frac{d}{dt}\int\left[\vec{g}_{mech} + \epsilon_0 (\vec{E}\times\vec{B})\right] d\tau$$

$$= \int_V \left[\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}(\vec{B}\times(\vec{\nabla}\times\vec{B})) + \epsilon_0 \vec{E}\times\partial\vec{B}/\partial t\right] d\tau.$$

$$= \int_S \overleftrightarrow{T}\cdot d\vec{s} \qquad \text{——— (3)}$$

where $\overleftrightarrow{T}$ is a tensor ~~called~~ called the Maxwell stress tensor. Physically this represents force per unit area acting on the surface.

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The 2nd term on the left of the eq^n (3)

$\int \epsilon_0 (\vec{E}\times\vec{B})\,d\tau$, represents a momentum and is called the momentum of em. the field. It does not contain any mass.

So the vector

$$\vec{g}_{em} = \epsilon_0(\vec{E}\times\vec{B})$$
$$= \vec{S}/c^2, \quad \text{where } c^2 = \frac{1}{\mu_0\epsilon_0}$$

is called the em momentum density.

So the pointing vector $\vec{S} = \left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)$. appears in dual role in an emr. field,

i) In ~~Carring~~ Carrying Energy

ii) In Carrying momentum.

So em field carries a momentum even in vaccum containing no charge and current. ~~The above~~ equation (3) in the general ~~sat~~ ~~state~~ statement of conservation of momentum in electrodynamics. Even a static field carry momentum as long as $\vec{E}\times\vec{B} \neq 0$.

8.2 Momentum

8.2.1 Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, **E** still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic

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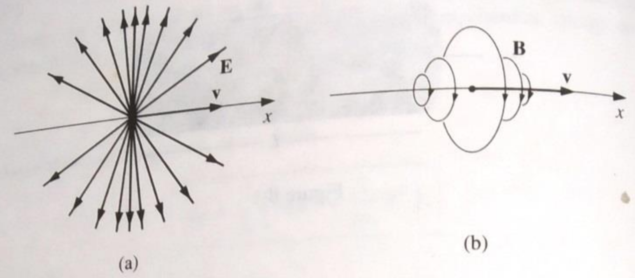

Figure 8.2

field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that **B** still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis. Of course, the electromagnetic force between them would tend to drive them off the axes, but let's assume that they're mounted on tracks, or something, so they're forced to maintain the same direction and the same speed (Fig. 8.3). The electric force between them is repulsive, but how about the magnetic force? Well, the magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward*. *The electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's third law.* In electro*statics* and magneto*statics* the third law holds, but in electro*dynamics* it does *not*.

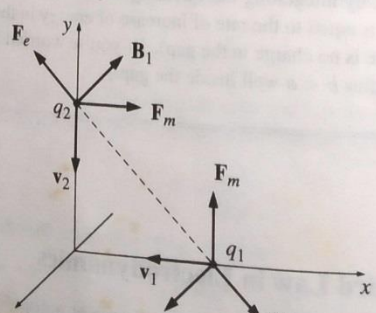

Figure 8.3

Well, that's an interesting curiosity, but then, how often does one actually use the third law, in practice? *Answer:* All the time! For the proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. When you tamper with the third law, you are placing conservation of momentum in jeopardy, and there is no principle in physics more sacred than *that*.

Momentum conservation is rescued in electrodynamics by the realization that *the fields themselves carry momentum*. This is not so surprising when you consider that we have already attributed *energy* to the fields. In the case of the two point charges in Fig. 8.3, whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum of the charges is momentum conservation restored. You'll see how this works out quantitatively in the following sections.

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8.2.3 Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}.$$

Equation 8.22 can therefore be written in the form

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}}\mathbf{S}\,d\tau + \oint_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}, \tag{8.28}$$

where $\mathbf{p}_{\text{mech}}$ is the total (mechanical) momentum of the particles contained in the volume $\mathcal{V}$. This expression is similar in structure to Poynting's theorem (Eq. 8.9), and it invites an analogous interpretation: The first integral represents *momentum stored in the electromagnetic fields themselves*:

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0\int_{\mathcal{V}}\mathbf{S}\,d\tau, \tag{8.29}$$

while the second integral is the *momentum per unit time flowing in through the surface*. Equation 8.28 is the general statement of *conservation of momentum* in electrodynamics: Any increase in the *total* momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields. (If $\mathcal{V}$ is *all* of space, then *no* momentum flows in or out, and $\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{em}}$ is constant.)

As in the case of conservation of charge and conservation of energy, conservation of momentum can be given a differential formulation. Let $\boldsymbol{\wp}_{\text{mech}}$ be the density of *mechanical* momentum, and $\boldsymbol{\wp}_{\text{em}}$ the density of momentum in the fields:

$$\boxed{\boldsymbol{\wp}_{\text{em}} = \mu_0\epsilon_0\mathbf{S}.} \tag{8.30}$$

Then Eq. 8.28, in differential form, says

$$\boxed{\frac{\partial}{\partial t}(\boldsymbol{\wp}_{\text{mech}} + \boldsymbol{\wp}_{\text{em}}) = \nabla\cdot\overleftrightarrow{\mathbf{T}}.} \tag{8.31}$$

Evidently $-\overleftrightarrow{\mathbf{T}}$ is the **momentum flux density**, playing the role of $\mathbf{J}$ (current density) in the continuity equation, or $\mathbf{S}$ (energy flux density) in Poynting's theorem. Specifically, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time. Notice that the Poynting vector has appeared in two quite different roles: $\mathbf{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\mathbf{S}$ is the momentum per unit volume stored in those fields. Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (the momentum current density) transported by the fields.

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8.2.4 Angular Momentum

By now the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.13)

$$u_{\mathrm{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right), \tag{8.32}$$

and *momentum* (Eq. 8.30)

$$\boldsymbol{\wp}_{\mathrm{em}} = \mu_0\epsilon_0 \mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}), \tag{8.33}$$

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell}_{\mathrm{em}} = \mathbf{r}\times\boldsymbol{\wp}_{\mathrm{em}} = \epsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]. \tag{8.34}$$

[3] See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994).
